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Equations relating structure functions of all orders

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Exact equations are given that relate velocity structure functions of arbitrary order with other statistics. 'Exact' means that no approximations are used except that the Navier–Stokes equation and incompressibility condition are assumed to be accurate. The exact equations are used to determine the structure function equations of all orders for locally homogeneous but anisotropic turbulence as well as for the locally isotropic case. The uses of these equations for investigating the approach to local homogeneity as well as to local isotropy and the balance of the equations and identification of scaling ranges are discussed. The implications for scaling exponents and investigation of intermittency are briefly discussed.

1. Introduction

Kolmogorov's (1941) equation and Yaglom's equation were the first two equations of the 'dynamic theory' of the local structure of turbulence. The name 'dynamic theory' was originated by Monin & Yaglom (1975) (their § 22) to mean the derivation of equations relating structure functions by use of the Navier–Stokes equation and/or the scalar conservation equation, and the investigation of the resulting statistical equations. Monin & Yaglom (1975) pointed out that the dynamic theory gives important relationships among structure functions, and that these relationships provide extensions of predictions based on dimensional analysis. Theoretical studies (Lindborg 1996; Hill 1997*a*) clarified the assumptions that are the basis of Kolmogorov's equation and give equations that are valid for anisotropic and locally homogeneous turbulence as well as for the case of local isotropy and local homogeneity. Antonia, Chambers & Browne (1983) and Chambers & Antonia (1984) used experimental data to study of the balance of the classic equations of Kolmogorov and Yaglom. There is renewed interest in examining the balance of those equations using both experimental and DNS data and in generalizing the equations to cases of inhomogeneous, non-stationary, and anisotropic turbulence (Lindborg 1999; Danaila *et al.* 1999*a*, *b*, *c*; Antonia *et al.* 2000). Whereas Kolmogorov's (1941) equation relates second- and third-order velocity structure functions, the next-order dynamic equation relates third- and fourth-order structure functions and a pressure-gradient, velocity–velocity structure function. The balance of that next-order equation has been examined by means of experimental and DNS data; this showed the behavior of the pressure-gradient, velocity–velocity structure function (Hill & Boratav 2001). There is now interest in dynamic-theory equations of arbitrarily high order N (Yakhot 2001). Such equations relate velocity structure functions of order N and $N+1$ and other statistics. Those equations are given in this paper.

Using the assumptions of local homogeneity, local isotropy and the Navier–Stokes equation, Yakhot (2001) derived the equation for the characteristic function of the probability distribution of two-point velocity differences. He uses that equation to derive higher-order dynamic equations. Equations for arbitrarily high-order structure functions can be obtained by repeated application of his differentiation procedure. Yakhot (2001) studies the inertial range, deduces a closure, and thereby determines the inertial-range scaling exponents of velocity structure functions. Yakhot's study is the first to make significant use of dynamic-theory equations to determine scaling exponents.

The purposes and theoretical method of the present paper differ from those of Yakhot (2001), but one purpose is to verify Yakhot's equations from our distinctly different derivation. That verification is given in § 5. In § 2, exact statistical equations relating velocity structure functions of any order are derived from the Navier-Stokes equation. 'Exact' means that no assumptions are made other than the assumption that the Navier–Stokes equation and incompressibility are accurate. Since the equations are exact, they apply to any flow, including laminar flow and inhomogeneous and anisotropic turbulent flow. The exact statistical equations can be used to verify DNS computations and detect their limitations. New experimental methods of Dahm and colleagues (Su & Dahm 1996) can also be tested. For example, if DNS data are used to evaluate the exact statistical equations, then the equations should balance to within numerical precision, otherwise a computational problem is indicated. In § 3, statistical equations valid for locally homogeneous and anisotropic turbulence are obtained from the exact equations; those equations can be used with DNS or experimental (Su & Dahm 1996) data to study the approach to local homogeneity of a particular flow. This can be done by quantifying the terms that are neglected when passing from exact equations to the locally homogeneous case, and by quantifying changes in the retained terms as local homogeneity is approached when the spatial separation vector is decreased. In § 4, statistical equations valid for locally isotropic and locally homogeneous turbulence are obtained from those for the locally homogeneous case. The approach to local isotropy can be studied by means analogous to the above described evaluation of local homogeneity. Such studies might shed light on the observed persistence of anisotropy (Pumir & Shraiman 1995; Shen & Warhaft 2000). All dynamic-theory equations are now available to extend the above-mentioned previous studies of the balance of dynamic-theory equations.

There have been many studies of the possibility that the inertial-range scaling exponents of structure-function components are unequal (e.g. Chen *et al.* 1997; Boratav & Pelz 1997; Boratav 1997; Grossmann, Lohse & Reeh 1997; van de Water & Herweijer 1999; Camussi & Benzi 1997; Dhruva, Tsuji & Sreenivasan 1997; Antonia, Zhou & Zhu 1998; Kahaleras, Malecot & Gagne 1996; Noullez *et al.* 1997; Nelkin 1999; Zhou & Antonia 2000; Kerr, Meneguzzi & Gotoh 2001). The usefulness of applying the higher-order dynamic-theory equations to those investigations is considered in § 6.

Derivation of the equations produces substantial mathematical detail. Matrix-based algorithms are invented such that the isotropic formulas for the divergence and Laplacian of isotropic tensors of any order can be generated by computer. The details of this mathematics are available and are herein referred to as the Archive.†

† The document 'Mathematics of structure-function equations of all orders' by R. J. Hill is available from the archive of the Journal of Fluid Mechanics in the Editorial Office at Cambridge and at xxx.lanl.gov/physics/0102055.

2. Exact two-point equations

The Navier–Stokes equation for velocity component $u_i(\boldsymbol{x},t)$ is

$$\partial_t u_i(\boldsymbol{x},t) + u_n(\boldsymbol{x},t)\partial_{x_n} u_i(\boldsymbol{x},t) = -\partial_{x_i} p(\boldsymbol{x},t) + \nu \partial_{x_n}\partial_{x_n} u_i(\boldsymbol{x},t), \tag{2.1}$$

and the incompressibility condition is $\partial_{x_n} u_n(\boldsymbol{x},t) = 0$. In (2.1), $p(\boldsymbol{x},t)$ is the pressure divided by the density (density is constant), ν is kinematic viscosity, and ∂ denotes partial differentiation with respect to its subscript variable. Summation is implied by repeated Roman indices. Consider another point $\boldsymbol{x}'$ such that $\boldsymbol{x}'$ and $\boldsymbol{x}$ are independent variables. For brevity, let $u_i = u_i(\boldsymbol{x},t)$, $u_i' = u_i(\boldsymbol{x}',t)$, etc. Require that $\boldsymbol{x}$ and $\boldsymbol{x}'$ have no relative motion. Then $\partial_{x_i} u_j' = 0$, $\partial_{x_i'} u_j = 0$, etc., and ∂_t is performed with both $\boldsymbol{x}$ and $\boldsymbol{x}'$ fixed. The change of independent variables from $\boldsymbol{x}$ and $\boldsymbol{x}'$ to the sum and difference independent variables is

$$\boldsymbol{X} \equiv (\boldsymbol{x}+\boldsymbol{x}')/2 \quad \text{and} \quad \boldsymbol{r} \equiv \boldsymbol{x}-\boldsymbol{x}', \quad \text{and define } r \equiv |\boldsymbol{r}|. \tag{2.2}$$

The relationship between the partial derivatives is

$$\partial_{x_i} = \partial_{r_i} + \tfrac{1}{2}\partial_{X_i}, \quad \partial_{x_i'} = -\partial_{r_i} + \tfrac{1}{2}\partial_{X_i}, \quad \partial_{X_i} = \partial_{x_i} + \partial_{x_i'}, \quad \partial_{r_i} = \tfrac{1}{2}(\partial_{x_i} - \partial_{x_i'}). \tag{2.3}$$

The change of variables organizes the equations in a revealing way because of the following properties. In the case of homogeneous turbulence, ∂_{X_i} operating on a statistic produces zero because that derivative is the rate of change with respect to the place where the measurement is performed. Consider a term in an equation composed of ∂_{X_i} operating on a statistic. For locally homogeneous turbulence, that term becomes negligible as r is decreased relative to the integral scale. For the homogeneous and locally homogeneous cases, the statistical equations retain their dependence on $\boldsymbol{r}$, which is the displacement vector of two points of measurement. Subtracting (2.1) at $\boldsymbol{x}'$ from (2.1) at $\boldsymbol{x}$, performing the change of variables (2.2), and using (2.3) gives

$$\partial_t v_i + U_n\partial_{X_n} v_i + v_n\partial_{r_n} v_i = -P_i + \nu(\partial_{x_n}\partial_{x_n} v_i + \partial_{x_n'}\partial_{x_n'} v_i), \tag{2.4}$$

where

$$v_i \equiv u_i - u_i', \quad U_n \equiv (u_n + u_n')/2, \quad P_i \equiv \partial_{x_i} p - \partial_{x_i'} p'. \tag{2.5}$$

Now multiply (2.4) by the product $v_j v_k \cdots v_l$ which contains $N-1$ factors of velocity difference, each factor having a distinct index. Sum the N equations as required to produce symmetry under interchange of each pair of indices, excluding the summation index n. Braces, i.e. $\{\circ\}$, denote the sum of all terms of a given type that produce symmetry under interchange of each pair of indices. The differentiation chain rule gives

$$\{v_j v_k \cdots v_l \partial_t v_i\} = \partial_t(v_j v_k \cdots v_l v_i), \tag{2.6}$$

$$\{v_j v_k \cdots v_l U_n \partial_{X_n} v_i\} = U_n\partial_{X_n}(v_j v_k \cdots v_l v_i) = \partial_{X_n}(U_n v_j v_k \cdots v_l v_i), \tag{2.7}$$

$$\{v_j v_k \cdots v_l v_n \partial_{r_n} v_i\} = v_n\partial_{r_n}(v_j v_k \cdots v_l v_i) = \partial_{r_n}(v_n v_j v_k \cdots v_l v_i). \tag{2.8}$$

The right-most expressions in (2.7) and (2.8) follow from the incompressibility property obtained from (2.3) and the fact that $\partial_{x_i} u_j' = 0$, $\partial_{x_i'} u_j = 0$; namely, $\partial_{X_n} U_n = 0$, $\partial_{X_n} v_n = 0$, $\partial_{r_n} U_n = 0$, $\partial_{r_n} v_n = 0$. The viscous term in equation (2.4) produces $\nu\{v_j v_k \cdots v_l(\partial_{x_n}\partial_{x_n} v_i + \partial_{x_n'}\partial_{x_n'} v_i)\}$; this expression is treated in the Archive. Thereby

$$\begin{aligned}\partial_t(v_j \cdots v_i) &+ \partial_{X_n}(U_n v_j \cdots v_i) + \partial_{r_n}(v_n v_j \cdots v_i)\\ &= -\{v_j \cdots v_l P_i\} + 2\nu\left[(\partial_{r_n}\partial_{r_n} + \tfrac{1}{4}\,\partial_{X_n}\partial_{X_n})(v_j \cdots v_i) - \{v_k \cdots v_l e_{ij}\}\right],\end{aligned} \tag{2.9}$$

where

$$e_{ij} \equiv (\partial_{x_n} u_i)(\partial_{x_n} u_j) + (\partial_{x'_n} u'_i)(\partial_{x'_n} u'_j) = (\partial_{x_n} v_i)(\partial_{x_n} v_j) + (\partial_{x'_n} v_i)(\partial_{x'_n} v_j). \tag{2.10}$$

The quantity $\{v_j \cdots v_l P_i\}$ can be expressed differently on the basis that (2.3) allows P_i to be written as $P_i = \partial_{X_i}(p - p')$. The derivation is in the Archive; the alternative formula is

$$\{v_j v_k \cdots v_l P_i\} = \{\partial_{X_i}[v_j v_k \cdots v_l (p - p')]\} - (N-1)(p - p')\{(s_{ij} - s'_{ij}) v_k \cdots v_l\}, \tag{2.11}$$

where the rate of strain tensor s_{ij} is defined by $s_{ij} \equiv (\partial_{x_i} u_j + \partial_{x_j} u_i)/2$.

2.1. *Hierarchy of exact statistical equations*

Consider the ensemble average because it commutes with temporal and spatial derivatives. The above notation of explicit indices is burdensome. Because the tensors are symmetric, it suffices to show only the number of indices. Define the following statistical tensors which are symmetric under interchange of any pair of indices, excluding the summation index n in the definition of $\boldsymbol{F}_{[N+1]}$:

$$\begin{aligned} \boldsymbol{D}_{[N]} &\equiv \langle v_j \cdots v_i \rangle, \quad \boldsymbol{F}_{[N+1]} \equiv \langle U_n v_j \cdots v_i \rangle, \\ \boldsymbol{T}_{[N]} &\equiv \langle \{v_j \cdots v_l P_i\} \rangle, \quad \boldsymbol{E}_{[N]} \equiv \langle \{v_k \cdots v_l e_{ij}\} \rangle, \end{aligned} \tag{2.12}$$

where angle brackets $\langle \ \rangle$ denote the ensemble average, and the subscripts N and $N+1$ within square brackets denote the number of indices. The left-hand side of each definition in (2.12) is in implicit-index notation for which only the number of indices is given; the right-hand sides in (2.12) are in explicit-index notation. The argument list for each tensor is understood to be $(\boldsymbol{X}, \boldsymbol{r}, t)$. The ensemble average of (2.9) is

$$\partial_t \boldsymbol{D}_{[N]} + \nabla_{\boldsymbol{X}} \cdot \boldsymbol{F}_{[N+1]} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]} = -\boldsymbol{T}_{[N]} + 2\nu [(\nabla_{\boldsymbol{r}}^2 + \tfrac{1}{4}\nabla_{\boldsymbol{X}}^2)\boldsymbol{D}_{[N]} - \boldsymbol{E}_{[N]}], \tag{2.13}$$

where $\nabla_{\boldsymbol{X}} \cdot \boldsymbol{F}_{[N+1]} \equiv \partial_{X_n} \langle U_n v_j \cdots v_i \rangle$, $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]} \equiv \partial_{r_n} \langle v_n v_j \cdots v_i \rangle$, $\nabla_{\boldsymbol{r}}^2 \equiv \partial_{r_n} \partial_{r_n}$, $\nabla_{\boldsymbol{X}}^2 \equiv \partial_{X_n} \partial_{X_n}$. The notations $\nabla_{\boldsymbol{X}} \cdot$, $\nabla_{\boldsymbol{X}}^2$, $\nabla_{\boldsymbol{r}} \cdot$, and $\nabla_{\boldsymbol{r}}^2$ are the divergence and Laplacian operators in $\boldsymbol{X}$-space and $\boldsymbol{r}$-space, respectively.

3. Homogeneous and locally homogeneous turbulence

Consider homogeneous turbulence and locally homogeneous turbulence; the latter applies for small r and large Reynolds number. The variation of the statistics with the location of measurement or of evaluation is zero for the homogeneous case and is neglected for the locally homogeneous case. Since that location is $\boldsymbol{X}$, the result of $\nabla_{\boldsymbol{X}}$ operating on a statistic vanishes or is neglected as the case may be. Thus the terms $\nabla_{\boldsymbol{X}} \cdot \boldsymbol{F}_{[N+1]}$ and $\frac{1}{4}\nabla_{\boldsymbol{X}}^2 \boldsymbol{D}_{[N]}$ are deleted in (2.13); then (2.13) becomes

$$\partial_t \boldsymbol{D}_{[N]} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]} = -\boldsymbol{T}_{[N]} + 2\nu [\nabla_{\boldsymbol{r}}^2 \boldsymbol{D}_{[N]} - \boldsymbol{E}_{[N]}]. \tag{3.1}$$

Because the $\boldsymbol{X}$-dependence is deleted, the argument list for each tensor is understood to be $(\boldsymbol{r}, t)$. Note that $\partial_t \boldsymbol{D}_{[N]}$ is not necessarily negligible for homogeneous turbulence. The ensemble average of equation (2.11) contains $\langle \{\partial_{X_i}[v_j v_k \cdots v_l (p - p')]\} \rangle = \{\partial_{X_i} \langle v_j v_k \cdots v_l (p - p') \rangle\} = \{0\} = 0$. Thus, (2.11) gives the alternative that

$$\boldsymbol{T}_{[N]} = -(N-1) \langle (p - p')\{(s_{ij} - s'_{ij}) v_k \cdots v_l\} \rangle. \tag{3.2}$$

One distinction between (3.1) and the hierarchy equations given in equations (13) and (17) by Arad, L'vov & Proccacia (1999) is that their t- and $\boldsymbol{r}$-derivatives operate on only one velocity difference within their product of such differences, whereas the derivatives in (2.9) and thus in (3.1) operate on all N of the velocity differences.

4. Isotropic and locally isotropic turbulence

Consider isotropic turbulence and locally isotropic turbulence; the latter applies for small r and large Reynolds number. Locally isotropic flows are a subset of locally homogeneous flows (Monin & Yaglom 1975, § 13.3) and similarly for the relationship between isotropic and homogeneous flows. Thus, the dynamical equations for locally isotropic and isotropic turbulence are obtained from (3.1) such that the variable $\boldsymbol{X}$ and the term $\nabla_X \cdot \boldsymbol{F}_{[N+1]}$ (see 2.13) do not appear. The tensors $\boldsymbol{D}_{[N]}$, $\boldsymbol{T}_{[N]}$, and $\boldsymbol{E}_{[N]}$ in (2.12) obey the isotropic formula. The Kronecker delta δ_{ij} is defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Let $\boldsymbol{\delta}_{[2P]}$ denote the product of P Kronecker deltas having $2P$ distinct indices, and let $\boldsymbol{W}_{[N]}(\boldsymbol{r})$ denote the product of N factors r_i/r each with a distinct index; the argument $\boldsymbol{r}$ is omitted when clarity does not suffer. Because each tensor in (2.12) is symmetric under interchange of any two indices, their isotropic formulas are particularly simple. Each formula is a sum of $M + 1$ terms, where $M = N/2$ if N is even, and $M = (N - 1)/2$ if N is odd. Each term is the product of a distinct scalar function with a $\boldsymbol{W}_{[N]}$ and a $\boldsymbol{\delta}_{[2P]}$. From one term to the next a pair of indices is transferred from a $\boldsymbol{W}_{[N]}$ to a $\boldsymbol{\delta}_{[2P]}$; examples are in the Archive. For the tensor $\boldsymbol{D}_{[N]}$, denote the Pth scalar function by $D_{N,P}(r, t)$. The isotropic formula for $\boldsymbol{D}_{[N]}$ is

$$\boldsymbol{D}_{[N]}(\boldsymbol{r}, t) = \sum_{P=0}^{M} D_{N,P}(r, t)\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}, \tag{4.1}$$

and the isotropic formulas for $\boldsymbol{T}_{[N]}$ and $\boldsymbol{E}_{[N]}$ have the analogous notation. Recall from § 2 the meaning of the notation $\{\circ\}$ whereby $\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}$ denotes the sum of all terms of the type $\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}$ that produce symmetry under interchange of each pair of indices. An example is $\{\boldsymbol{W}_{[1]}(\boldsymbol{r})\boldsymbol{\delta}_{[2]}\} = (r_k/r)\delta_{ij} + (r_j/r)\delta_{ki} + (r_i/r)\delta_{jk}$.

A special Cartesian coordinate system simplifies the isotropic formulas. This coordinate system has the positive 1-axis parallel to the direction of $\boldsymbol{r}$, and the 2- and 3-axes are therefore perpendicular to $\boldsymbol{r}$. Let N_1, N_2, and N_3 be the number of indices of a component of $\boldsymbol{D}_{[N]}$ that are 1, 2, and 3, respectively; such that $N = N_1 + N_2 + N_3$. Because of symmetry, the order of indices is immaterial so that a component of $\boldsymbol{D}_{[N]}$ can be identified by N_1, N_2, and N_3. Thus, denote a component of $\boldsymbol{D}_{[N]}$ by $D_{[N_1,N_2,N_3]}$ which is a function of $\boldsymbol{r}$ and t. Likewise, $\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}_{[N_1,N_2,N_3]}$ is a specific component of the tensor $\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}$. If, in (4.1) N_1 of the indices are assigned the value 1, and N_2 and N_3 of the indices are assigned the values 2 and 3, respectively, then $D_{[N_1,N_2,N_3]}$ and $\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}_{[N_1,N_2,N_3]}$ will appear on the left-hand and right-hand sides of (4.1), respectively. The $\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}_{[N_1,N_2,N_3]}$ are numerical coefficients that do not depend on $\boldsymbol{r}$ because $r_1/r = r/r = 1$, $r_2/r = r_3/r = 0$. From the Archive, the values of the coefficients are

$$\text{if } 2P < N_2 + N_3 \quad \text{then } \{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}_{[N_1,N_2,N_3]} = 0, \quad \text{otherwise,} \tag{4.2}$$

$$\{\boldsymbol{W}_{[N-2P]}(\boldsymbol{r})\boldsymbol{\delta}_{[2P]}\}_{[N_1,N_2,N_3]} = N_1!N_2!N_3! \Big/ \left[(N-2P)!2^P \left(\frac{N_2}{2}\right)! \left(\frac{N_3}{2}\right)! \left(P - \frac{N_2}{2} - \frac{N_3}{2}\right)!\right]. \tag{4.3}$$

By applying (4.1)–(4.3) for all combinations of indices, one can determine which components $D_{[N_1,N_2,N_3]}$ are zero and which are non-zero, identify $M + 1$ linearly independent equations that determine the $D_{N,P}$ in terms of $M+1$ of the $D_{[N_1,N_2,N_3]}$, and find algebraic relationships between the remaining non-zero $D_{[N_1,N_2,N_3]}$. The derivations are in the Archive; a summary follows.

A component $D_{[N_1,N_2,N_3]}$ is non-zero only if both N_2 and N_3 are even, and therefore N_1 is odd if N is odd, and N_1 is even if N is even. Thereby, $(M+1)(M+2)/2$ components are non-zero. There are 3^N components of $\boldsymbol{D}_{[N]}$; thus the other $3^N-(M+1)(M+2)/2$ components are zero.

There exist $(M+1)M/2$ kinematic relationships among the non-zero components of $\boldsymbol{D}_{[N]}$. For each of the $M+1$ cases of N_1, these relationships are expressed by the proportionality

$$\begin{aligned} &D_{[N_1,2L,0]} : D_{[N_1,2L-2,2]} : D_{[N_1,2L-4,4]} : \cdots : D_{[N_1,0,2L]} \\ &\quad = [(2L)!0!/L!0!] : [(2L-2)!2!/(L-1)!1!] : \\ &\qquad [(2L-4)!4!/(L-2)!2!] : \cdots : [0!(2L)!/0!L!]. \end{aligned} \tag{4.4}$$

For $N=4$ with $L=2$ (4.4) gives $D_{[0,4,0]} : D_{[0,2,2]} : D_{[0,0,4]} = 12 : 4 : 12$. In explicit-index notation this can be written as $D_{2222} = 3D_{2233} = D_{3333}$, which was discovered by Millionshtchikov (1941) and is the only previously known such relationship. Now, all such relationships are known from (4.4).

There remain $M+1$ linearly independent non-zero components of $\boldsymbol{D}_{[N]}$. This must be so because there are $M+1$ terms in (4.1) and the $M+1$ scalar functions $D_{N,P}$ therein must be related to $M+1$ components. Consider the $M+1$ linearly independent equations that determine the $D_{N,P}$ in terms of $M+1$ of the $D_{[N_1,N_2,N_3]}$. For simplicity, the chosen components can all have $N_3=0$, i.e. the choice of linearly independent components can be $D_{[N,0,0]}, D_{[N-2,2,0]}, D_{[N-4,4,0]}, \ldots, D_{[N-2M,2M,0]}$. As described above, assigning index values in (4.1) results in the chosen components on the left-hand side and algebraic expressions on the right-hand side that contain the coefficients (4.2)–(4.3). In the Archive, those equations are expressed in matrix form and solved by matrix inversion methods. Given experimental or DNS data or a theoretical formula for the chosen components, the solution of the algebraic equations determines the functions $D_{N,P}$ in (4.1); then (4.1) completely specifies the tensor $\boldsymbol{D}_{[N]}$.

The matrix algorithm in the Archive is an efficient means of determining isotropic expressions for the terms $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]}$ and $\nabla_{\boldsymbol{r}}^2 \boldsymbol{D}_{[N]}$ in (3.1). From the example for $N=2$ in the Archive, (3.1) gives the two scalar equations

$$\begin{aligned} &\partial_t D_{11} + \left(\partial_r + \frac{2}{r}\right) D_{111} - \frac{4}{r} D_{122} \\ &\quad = -T_{11} + 2\nu \left[\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2}\right) D_{11} + \frac{4}{r^2} D_{22}\right] - 2\nu E_{11} \\ &\quad = 2\nu \left[\partial_r^2 D_{11} + \frac{2}{r}\partial_r D_{11} + \frac{4}{r^2}(D_{22} - D_{11})\right] - 4\varepsilon/3, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \partial_t D_{22} + \left(\partial_r + \frac{4}{r}\right) D_{122} &= -T_{22} + 2\nu \left[\frac{2}{r^2} D_{11} + \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2}\right) D_{22}\right] - 2\nu E_{22} \\ &= 2\nu \left[\partial_r^2 D_{22} + \frac{2}{r}\partial_r D_{22} - \frac{2}{r^2}(D_{22} - D_{11})\right] - 4\varepsilon/3, \end{aligned} \tag{4.6}$$

where use was made of the fact (Hill 1997*a*) that local isotropy gives $T_{11} = T_{22} = 0$ and $2\nu E_{11} = 2\nu E_{22} = 4\varepsilon/3$, where ε is the average energy dissipation rate. Since (4.5)–(4.6) are the same as equations (43)–(44) of Hill (1997*a*), and since Hill (1997*a*) shows how these equations lead to Kolmogorov's equation and his 4/5 law, further discussion of (4.5)–(4.6) is unnecessary. From the example for $N = 3$ in the Archive,

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu [(\nabla_r^2 \boldsymbol{D})_{111} - E_{111}], \tag{4.7}$$

$$\partial_t D_{122} + \left(\partial_r + \frac{4}{r}\right) D_{1122} - \frac{4}{3r} D_{2222} = -T_{122} + 2\nu [(\nabla_r^2 \boldsymbol{D})_{122} - E_{122}], \tag{4.8}$$

$$(\nabla_r^2 \boldsymbol{D})_{111} \equiv \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{6}{r^2}\right) D_{111} + \frac{12}{r^2} D_{122} = \left(-\frac{4}{r^2} + \frac{4}{r}\partial_r + \partial_r^2\right) D_{111}, \tag{4.9}$$

$$(\nabla_r^2 \boldsymbol{D})_{122} \equiv \frac{2}{r^2} D_{111} + \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{8}{r^2}\right) D_{122} = \frac{1}{6}\left(\frac{4}{r^2} - \frac{4}{r}\partial_r + 5\partial_r^2 + r\partial_r^3\right) D_{111}. \tag{4.10}$$

The incompressibility condition, $D_{122} = \frac{1}{6}(D_{111} + r\partial_r D_{111})$, was used in (4.9)–(4.10). Since Hill & Boratav (2001) discuss these equations and evaluate them using data, further discussion of (4.7)–(4.10) is unnecessary.

The terms $\partial_t \boldsymbol{D}_{[N]}$, $-\boldsymbol{T}_{[N]}$, and $-2\nu \boldsymbol{E}_{[N]}$ in (3.1) have a repetitive structure in the isotropic equations, e.g. for $N = 4$ the three equations are

$$\partial_t D_{1111} + (\nabla_r \cdot \boldsymbol{D}_{[5]})_{1111} = -T_{1111} + 2\nu [(\nabla_r^2 \boldsymbol{D}_{[4]})_{1111} - E_{1111}], \tag{4.11}$$

$$\partial_t D_{1122} + (\nabla_r \cdot \boldsymbol{D}_{[5]})_{1122} = -T_{1122} + 2\nu [(\nabla_r^2 \boldsymbol{D}_{[4]})_{1122} - E_{1122}], \tag{4.12}$$

$$\partial_t D_{2222} + (\nabla_r \cdot \boldsymbol{D}_{[5]})_{2222} = -T_{2222} + 2\nu [(\nabla_r^2 \boldsymbol{D}_{[4]})_{2222} - E_{2222}]. \tag{4.13}$$

Thus, it suffices to give the isotropic formulas for the divergence $\nabla_r \cdot \boldsymbol{D}_{[N+1]}$ and Laplacian $\nabla_r^2 \boldsymbol{D}_{[N]}$; for $N = 4$ to 7, those isotropic formulas are given in table 1. For $N = 4$ and 5 there are $M + 1 = 3$ equations; there are $M + 1 = 4$ equations for both $N = 6$ and 7.

5. Comparison with previous results

The expression $(\partial_r + 2/r)D_{111} - (4/r)D_{122}$ in (4.5) is the same as equation (9) of Yakhot (2001), and (41) of Hill (1997*a*). The expression $(\partial_r + 2/r)D_{1111} - (6/r)D_{1122}$ in (4.7) is the same as in the equation that follows Yakhot's equation (11), and in equation (16) of Hill & Boratav (2001) and in equation (8) of Kurien & Sreenivasan (2001); $(\partial_r + 4/r)D_{1122} - (4/3r)D_{2222}$ in (4.8) is the same as in equation (13) of Hill & Boratav (2001) and equation (10) of Kurien & Sreenivasan (2001). The expressions $(\partial_r + 2/r)D_{[6,0,0]} - (10/r)D_{[4,2,0]}$ and $(\partial_r + 6/r)D_{[2,4,0]} - (6/5r)D_{[0,6,0]}$ for the case $N = 5$ in table 1 are the same as in equations (9) and (10) of Kurien & Sreenivasan (2001). More generally, the isotropic formulas for $\nabla_r \cdot \boldsymbol{D}_{[N+1]}$ for the case $N_1 = N$, $N_2 = N_3 = 0$ are $(\partial_r + 2/r)D_{[N,0,0]} - [2(N-1)/r]D_{[N-2,2,0]}$ which agrees with the left-hand side of equation (7) of Yakhot (2001). The other components of $\nabla_r \cdot \boldsymbol{D}_{[N+1]}$ were not given by Yakhot (2001). The expressions from the Laplacian in (4.5)–(4.6) are the same as in (41)–(42) of Hill (1997*a*); and (4.9)–(4.10) are the same as (7)–(8) of Hill & Boratav (2001). All of the remaining results do not appear to have been given previously. The above comparisons are sufficient to verify the matrix algorithm for generating the structure-function equations to any desired order, as well as to independently validate the derivation of Yakhot (2001).

$N=4$	
$\left(\partial_r+\frac{2}{r}\right)D_{[5,0,0]}-\frac{8}{r}D_{[3,2,0]}$	$\left(\partial_r^2+\frac{2}{r}\partial_r-\frac{8}{r^2}\right)D_{[4,0,0]}+\frac{14}{r^2}D_{[2,2,0]}+\frac{10}{3r^2}D_{[0,4,0]}$
$\left(\partial_r+\frac{4}{r}\right)D_{[3,2,0]}-\frac{8}{3r}D_{[1,4,0]}$	$\frac{2}{r^2}D_{[4,0,0]}+\left(-\frac{52}{3r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[2,2,0]}+\frac{34}{9r^2}D_{[0,4,0]}$
$\left(\partial_r+\frac{6}{r}\right)D_{[1,4,0]}$	$\frac{2}{r^2}D_{[2,2,0]}+\left(-\frac{2}{3r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[0,4,0]}$
$N=5$	
$\left(\partial_r+\frac{2}{r}\right)D_{[6,0,0]}-\frac{10}{r}D_{[4,2,0]}$	$\left(\partial_r^2+\frac{2}{r}\partial_r-\frac{10}{r^2}\right)D_{[5,0,0]}-\frac{14}{r^2}D_{[3,2,0]}+\frac{54}{r^2}D_{[1,4,0]}$
$\left(\partial_r+\frac{4}{r}\right)D_{[4,2,0]}-\frac{4}{r}D_{[2,4,0]}$	$\frac{2}{r^2}D_{[5,0,0]}+\left(-\frac{154}{5r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[3,2,0]}+\frac{94}{5r^2}D_{[1,4,0]}$
$\left(\partial_r+\frac{6}{r}\right)D_{[2,4,0]}-\frac{6}{5r}D_{[0,6,0]}$	$\frac{6}{5r^2}D_{[3,2,0]}+\left(-\frac{16}{5r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[1,4,0]}$
$N=6$	
$\left(\partial_r+\frac{2}{r}\right)D_{[7,0,0]}-\frac{12}{r}D_{[5,2,0]}$	$\left(\partial_r^2+\frac{2}{r}\partial_r-\frac{12}{r^2}\right)D_{[6,0,0]}-\frac{108}{r^2}D_{[4,2,0]}+\frac{920}{3r^2}D_{[2,4,0]}-\frac{416}{15r^2}D_{[0,6,0]}$
$\left(\partial_r+\frac{4}{r}\right)D_{[5,2,0]}-\frac{16}{3r}D_{[3,4,0]}$	$\frac{2}{r^2}D_{[6,0,0]}+\left(-\frac{242}{5r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[4,2,0]}+\frac{824}{15r^2}D_{[2,4,0]}-\frac{248}{75r^2}D_{[0,6,0]}$
$\left(\partial_r+\frac{6}{r}\right)D_{[3,4,0]}-\frac{12}{5r}D_{[1,6,0]}$	$\frac{4}{5r^2}D_{[4,2,0]}+\left(-\frac{112}{15r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[2,4,0]}+\frac{4}{3r^2}D_{[0,6,0]}$
$\left(\partial_r+\frac{8}{r}\right)D_{[1,6,0]}$	$\frac{2}{3r^2}D_{[2,4,0]}+\left(-\frac{2}{15r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[0,6,0]}$
$N=7$	
$\left(\partial_r+\frac{2}{r}\right)D_{[8,0,0]}-\frac{14}{r}D_{[6,2,0]}$	$\left(\partial_r^2+\frac{2}{r}\partial_r-\frac{14}{r^2}\right)D_{[7,0,0]}-\frac{316}{r^2}D_{[5,2,0]}+\frac{3376}{3r^2}D_{[3,4,0]}-\frac{1376}{5r^2}D_{[1,6,0]}$
$\left(\partial_r+\frac{4}{r}\right)D_{[6,2,0]}-\frac{20}{3r}D_{[4,4,0]}$	$\frac{2}{r^2}D_{[7,0,0]}+\left(-\frac{1472}{21r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[5,2,0]}+\frac{7808}{63r^2}D_{[3,4,0]}-\frac{304}{15r^2}D_{[1,6,0]}$
$\left(\partial_r+\frac{6}{r}\right)D_{[4,4,0]}-\frac{18}{5r}D_{[2,6,0]}$	$\frac{4}{7r^2}D_{[5,2,0]}+\left(-\frac{206}{15r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[3,4,0]}+\frac{1132}{175r^2}D_{[1,6,0]}$
$\left(\partial_r+\frac{8}{r}\right)D_{[2,6,0]}-\frac{8}{7r}D_{[0,8,0]}$	$\frac{2}{7r^2}D_{[3,4,0]}+\left(-\frac{76}{35r^2}+\partial_r^2+\frac{2}{r}\partial_r\right)D_{[1,6,0]}$

TABLE 1. The isotropic formulas for $\nabla_r\cdot\boldsymbol{D}_{[N+1]}$ are on the left and those for $\nabla_r^2\boldsymbol{D}_{[N]}$ are on the right.

6. Summary and discussion

The third paragraph of the introduction summarizes part of this paper and is not repeated here. In addition: All of the kinematic relationships (4.4) between components of isotropic, symmetric structure functions of arbitrary order have been identified, whereas previously only one was known. All of the components that are zero have been identified (a recent experimental evaluation of some of them is given by Kurien & Sreenivasan 2000). The kinematic relationships show that the scaling exponents of certain different components must be equal; if the exponents are not equal when evaluated using one's data, then the kinematic relationships (4.4) provide a measure of either the error in the exponents or the deviation from local isotropy.

The dynamic equations of order N can be used to test the extent of a scaling range for evaluation of scaling exponents of velocity structure functions of order $N+1$ because the time-derivative and viscous terms should be zero in an inertial range. The graphical presentations of the balance of Kolmogorov's equation by Antonia *et al.* (1983), Chambers & Antonia (1984), Danaila *et al.* (1999*a*, *b*), and Antonia *et al.* (2000) show the extent of, or deviation from, inertial-range exponents. The higher-order equations given here can be used in an analogous manner.

The energy dissipation rate ε plays an essential role at all r in Kolmogorov's equation. In our formulation ε arises in (4.5)–(4.6) from the tensor components $2\nu E_{11}$ and $2\nu E_{22}$. On the other hand, for the next-order equations (4.7)–(4.8) Hill (1997*b*) showed that the corresponding terms $2\nu E_{111}$ and $2\nu E_{122}$ are negligible in the inertial range. Yakhot (2001) shows that the components $E_{[N,0,0]}$ are negligible in the inertial range for all of the higher-order equations for which N is odd. Kolmogorov's (1941) inertial-range scaling using ε and r as the only relevant parameters can be used to estimate the relative magnitudes of the term $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]}$ in (3.1) to the terms $2\nu\nabla_{\boldsymbol{r}}^2\boldsymbol{D}_{[N]}$ and $2\nu\boldsymbol{E}_{[N]}$. Doing so, the ratio of any non-zero component of $2\nu\nabla_{\boldsymbol{r}}^2\boldsymbol{D}_{[N]}$ or $2\nu\boldsymbol{E}_{[N]}$ to the corresponding component of $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]}$ is proportional to $\nu/r^{4/3}\varepsilon^{1/3} = (r/\eta)^{-4/3}$, which asymptotically vanishes in the inertial range ($\eta \equiv (\nu^3/\varepsilon)^{1/4}$). Thus, both terms $2\nu\nabla_{\boldsymbol{r}}^2\boldsymbol{D}_{[N]}$ and $2\nu\boldsymbol{E}_{[N]}$ are to be neglected in an inertial range if $N > 2$.

One concludes that all equations of order higher than Kolmogorov's equation reduce to the isotropic formula for $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{D}_{[N+1]} = -\boldsymbol{T}_{[N]}$ in the inertial range. This formula shows that $\boldsymbol{T}_{[N]}$ is at the heart of two issues that have received much attention: (i) whether or not different components of the velocity structure function $\boldsymbol{D}_{[N+1]}$ have differing exponents in the inertial range, and (ii) the increasing deviation of those exponents from Kolmogorov scaling as N increases. The physical basis for the importance of $\boldsymbol{T}_{[N]}$ is the importance of vortex tubes to the intermittency phenomenon (Pullin & Saffman 1998) combined with the fact that the pressure-gradient force is essential to the existence of vortex tubes; the pressure-gradient force prevents a vortex tube from cavitating despite the centrifugal force. Pressure gradients are the sinews of vortices. Direct investigation of $\boldsymbol{T}_{[N]}$ using DNS can reveal much about the two issues.

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REFERENCES

Antonia, R. A., Chambers, A. J. & Browne, L. W. B. 1983 Relations between structure functions of velocity and temperature in a turbulent jet. *Exps. Fluids* **1**, 213–219.

Antonia, R. A., Zhou, T., Danaila, L. & Anselmet, F. 2000 Streamwise inhomogeneity of decaying grid turbulence. *Phys. Fluids* **12**, 3086–3089.

Antonia, R. A., Zhou, T. & Zhu, Y. 1998 Three-component vorticity measurements in a turbulent grid flow. *J. Fluid Mech.* **374**, 29–57.

Arad, I., L'vov, V. S. & Procaccia, I. 1999 Correlation functions in isotropic and anisotropic turbulence: The role of the symmetry group. *Phys. Rev.* E **59**, 6753–6765.

Boratav, O. N. 1997 On recent intermittency models of turbulence. *Phys. Fluids* **9**, 1206–1208.

Boratav, O. N. & Pelz, R. B. 1997 Structures and structure functions in the inertial range of turbulence. *Phys. Fluids* **9**, 1400–1415.

Camussi, R. & Benzi, R. 1997 Hierarchy of transverse structure functions. *Phys. Fluids* **9**, 257–259.

Chambers, A. J. & Antonia, R. A. 1984 Atmospheric estimates of power-law exponents μ and μ_θ. *Boundary-Layer Met.* **28**, 343–352.

Chen, S., Sreenivasan, K. R., Nelkin, M. & Cao, N. 1997 Refined similarity hypothesis for transverse structure functions in fluid turbulence. *Phys. Rev. Lett.* **79**, 2253–2256.

Danaila, L., Anselmet, F., Zhou, T. & Antonia, R. A. 1999*a* A generalization of Yaglom's equation which accounts for the large-scale forcing in heated decaying turbulence. *J. Fluid Mech.* **391**, 359–372.

Danaila, L., Dusek, J., Le Gal, P., Anselmet, F., Brun, C. & Pumir, A. 1999*b* Planar isotropy of passive scalar turbulent mixing with a mean perpendicular gradient. *Phys. Rev.* E **60**, 1691–1707.

Danaila, L., Le Gal, P., Anselmet, F., Plaza, F. & Pinton, J. F. 1999*c* Some new features of the passive scalar mixing in a turbulent flow. *Phys. Fluids* **11**, 636–646.

Dhruva, B., Tsuji, Y. & Sreenivasan, K. R. 1997 Transverse structure functions in high-Reynolds-number turbulence. *Phys. Rev.* E **56**, R4928–R4930.

Grossmann, S., Lohse, D. & Reeh, A. 1997 Different intermittency for longitudinal and transversal turbulent fluctuations. *Phys. Fluids* **9**, 3817–3825.

Hill, R. J. 1997*a* Applicability of Kolmogorov's and Monin's equations of turbulence. *J. Fluid Mech.* **353**, 67–81.

Hill, R. J. 1997*b* Pressure-gradient, velocity–velocity structure function for locally isotropic turbulence in incompressible fluid. *NOAA Tech. Memo.* ERL ETL-277 (available from N.T.I.S., 5285 Port Royal Rd., Springfield, VA 22161).

Hill, R. J. & Boratav, O. N. 2001 Next-order structure-function equations. *Phys. Fluids* **13**, 276–283.

Kahaleras, H., Malecot, Y. & Gagne, Y. 1996 Transverse velocity structure functions in developed turbulence. In *Advances in Turbulence VI*, pp. 235–238. Kluwer.

Kerr, R. M., Meneguzzi, M. & Gotoh, T. 2001 An inertial range length scale in structure functions. *Phys. Fluids* (to appear); preprint xxx.lanl.gov/physics/0005004.

Kolmogorov, A. N. 1941 Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, 16.

Kurien, S. & Sreenivasan, K. R. 2000 Anisotropic scaling contributions to high-order structure functions in high-Reynolds-number turbulence. *Phys. Rev.* E **62**, 2206–2212.

Kurien, S. & Sreenivasan, K. R. 2001 Experimental results on new dynamical equations in three-dimensional turbulence.

Lindborg, E. 1996 A note on Kolmogorov's third-order structure-function law, the local isotropy hypothesis and the pressure–velocity correlation. *J. Fluid Mech.* **326**, 343–356.

Lindborg, E. 1999 Correction to the four-fifths law due to variations of the dissipation. *Phys. Fluids* **11**, 510–512.

Millionshtchikov, M. D. 1941 On the theory of homogeneous isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, 611–614.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics: Mechanics of Turbulence*, vol. 2. MIT Press.

Nelkin, M. 1999 Enstrophy and dissipation must have the same scaling exponent in high Reynolds number turbulence. *Phys. Fluids* **11**, 2202–2204.

Noullez, A., Wallace, G., Lempert, W., Miles, R. B. & Frisch, U. 1997 Transverse velocity increments in turbulent flow using the relief technique. *J. Fluid Mech.* **339**, 287–307.

Pullin, D. I. & Saffman, P. G. 1998 Vortex dynamics in turbulence. *Ann. Rev. Fluid Mech.* **30**, 31–51.

Pumir, A. & Shraiman, B. I. 1995 Persistent small scale anisotropy in homogeneous shear flows. *Phys. Rev. Lett.* **75**, 3114–3117.

Shen, X. & Warhaft, Z. 2000 The anisotropy of the small scale structure in high Reynolds number ($R_\lambda \sim 1000$) turbulent shear flow. *Phys. Fluids* **12**, 2976–2989.

Su, L. K. & Dahm, W. J. A. 1996 Scalar imaging velocimetry measurements of the velocity gradient tensor field in turbulent flows. I. Experimental results. *Phys. Fluids* **8**, 1883–1906.

Water, W. van de & Herweijer, J. A. 1999 Higher-order structure functions of turbulence. *J. Fluid Mech.* **387**, 3–37.

Yakhot, V. 2001 Mean-field approximation and a small parameter in turbulence theory. *Phys. Rev.* E **63**, 026307-1–026307-11.

Zhou, T. & Antonia, R. A. 2000 Reynolds number dependence of the small-scale structure of grid turbulence. *J. Fluid Mech.* **406**, 81–107.